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TORSIONAL VIBRATION OF SIZE-DEPENDENT VISCOELASTIC RODS USING NONLOCAL STRAIN AND VELOCITY GRADIENT THEORY

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Abstract. In this paper the torsional vibration of size-dependent viscoelastic nanorods embedded in an elastic medium with different boundary conditions is investigated. The novelty of this study consists of combining the nonlocal theory with the strain and velocity gradient theory to capture both softening and stiffening size-dependent behavior of the nanorods. The governing equation of motion and its boundary conditions for the vibration analysis of nanorods are derived by employing Hamilton's principle. It is shown that the expressions of the classical stress and the stress gradient resultants are only defined for different values of the nonlocal and strain gradient parameters. The case where these are equal may seem to result in an inconsistency to the general equation of motion and the related non-classical boundary conditions. A rigorous investigation is conducted to prove that that the proposed solution is consistent with physics. Damped eigenvalue solutions are obtained analytically and results of linear free vibration response are obtained for various length-scales.

1 INTRODUCTION

Over the past few decades the demand of nanomaterials has been increasing enormously in various applications like actuators, sensors, microscopes, micro/nano electro mechanical systems (MEMS)/(NEMS). Micro/Nano-scaled structures are made of structural elements which can be in the form of bars, rods, beams, plates or shell structures. Recently, several non-classical continuum theories that incorporate the effect of material length scales have been proposed in the literature to predict the behavior of nanostructures. These include nonlocal, gradient elasticity and couple stress theories or a combination of these theories.

Nonlocal theories assume that the stress at a point is not a function of the strain at that point but is a function of the strain in the entire domain containing the points [1]. Nonlocal elastic models can only model nanostructures exhibiting softening behavior which indicates that "smaller is more compliant" [2].

On the other hand, the gradient elasticity theory stipulates that nanostructures should be modeled as atoms with higher-order deformation mechanism and the total stress should account for some additional strain gradient terms [3]-[4]. Furthermore, gradient elasticity theory can only model nanostructures exhibiting hardening behavior which indicates that "smaller is stiffer" [2]. Therefore, it can be concluded that combining both theories allows the modeling of nanostructures exhibiting at the same time hardening and softening behavior. This has been confirmed experimentally through measurements on certain nanostructures [5].

Micro/nano rods subjected to torsional loads have been widely used in various types of MEMS/NEMS applications including torsional springs in NEMS oscillators [6], torsional micromirrors [7] and torsional microscanners [8]. Therefore, the accurate modeling of the static and dynamic torsional behavior of micro/nano bars seems to be essential in order to understand the mechanical behavior of these micro/nano systems. There have been few studies related to the size-dependent torsional vibration of nanotubes/nanorods. Most of these studies utilized the differential nonlocal model [9]-[15]. On the other hand, Kahrobaiyan et al. [16] used strain gradient theory to obtain closed-form analytical solutions for the static and free torsional vibration of a microbar.

A review of the works related to torsional vibration of nanorods revealed that they were either based on nonlocal elasticity theory or strain gradient theory and did not account for viscoelastic effects. The novelty of this paper consists of combining the nonlocal theory with the strain and velocity gradient theory to study the torsional vibration of a viscoelastic nanorod embedded in an elastic medium. This theory involves three length-scale parameters, namely, a nonlocal, a strain gradient and a velocity gradient parameter denoted, respectively, μ_0, l_s and l_k. It will be shown that the expressions of the classical stress and the stress gradient resultants are only defined when $\mu_0 \neq l_s$. The case $\mu_0 = l_s$ may seem to result in an inconsistency to the general equation of motion and the related non-classical boundary conditions. In fact, the expression of the stress gradient resultant may suggest an infinite value when $\mu_0 = l_s$ [17]. However, as an additional novelty of this work, it will be shown that calculating the limit of the stress gradient resultant is finite and, therefore, the proposed solution will not show any inconsistency.

2 NONLOCAL STRAIN GRADIENT VISCOELASTIC THEORY

The nonlocal strain gradient theory proposed by [18, 19] stipulates that the total stress tensor $\mathbf{t}$ accounts for both the nonlocal stress tensor σ and the higher-order strain gradient nonlocal stress tensor $\nabla\sigma^{(1)}$, in which $\sigma^{(1)}$ is the higher-order nonlocal stress tensor.

$$\mathbf{t} = \sigma - \nabla\sigma^{(1)} \tag{1}$$

where ∇ is the gradient operator and σ and $\sigma^{(\mathbf{1})}$ are given by

$$\sigma = \int_V \alpha_0\left(\left|\mathbf{x}-\mathbf{x}'\right|, e_0 a\right)\mathbf{C} : \varepsilon\left(\mathbf{x}'\right)\ dV \tag{2a}$$

$$\sigma^{(1)} = l_s^2 \int_V \alpha_1\left(\left|\mathbf{x}-\mathbf{x}'\right|, e_1 a\right)\mathbf{C} : \nabla\varepsilon\left(\mathbf{x}'\right)\ dV \tag{2b}$$

in which $\varepsilon\left(\mathbf{x}'\right)$ and $\nabla\varepsilon\left(\mathbf{x}'\right)$ are, respectively, the classical strain tensor and its gradient at point $\mathbf{x}'$, $\mathbf{C}$ is the fourth-order elasticity tensor, l_s is the strain gradient length-scale parameter, $e_0 a$ and $e_1 a$ are nonlocal parameters representing the significance of the inter-atomic long-range force, and α_0 and α_1 are kernel functions.

In view of the difficulty in using the integral constitutive relations (1), (2a) and (2b), Eringen [1] proposed an equivalent differential model. Thus, assuming $e_0 a = e_1 a = ea = \mu_0$ and for a suitable choice of the kernel functions α_0 and α_1, Eqs. (2a) and (2b) become

$$\left(1-\mu_0^2\nabla^2\right)\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon} \tag{3a}$$

$$\left(1-\mu_0^2\nabla^2\right)\boldsymbol{\sigma}^{(1)} = l_s^2\mathbf{C} : \nabla\boldsymbol{\varepsilon} \tag{3b}$$

where ∇^2 is the Laplacian operator. Substituting (3a) and (3b) into (1) yields

$$\left(1-\mu_0^2\nabla^2\right)\mathbf{t} = \left(1-l_s^2\nabla^2\right)\mathbf{C} : \boldsymbol{\varepsilon} \tag{4}$$

Furthermore, for a torsional rod-type structure defined in a cylindrical coordinate system (r, θ, x) where r is the radial axis, θ is the angular axis and x is the longitudinal axis, we assume the size-dependency is only accounted for in the longitudinal direction and neglected in the other directions. Therefore, Eq. (4) can be reduced to the following:

$$\left(1-\mu_0^2\frac{\partial^2}{\partial x^2}\right)t_{r\theta} = \left(1-l_s^2\frac{\partial^2}{\partial x^2}\right)G\varepsilon_{r\theta} \tag{5}$$

where ∇^2 was replaced by $\partial^2/\partial x^2$, $t_{r\theta}$ is the total shear stress, $\varepsilon_{r\theta}$ is the shear strain and G is the rod's modulus of rigidity. This model combines Eringen's nonlocal elasticity theory and strain gradient theory to obtain the Nonlocal Strain Gradient (NSG) theory. Viscoelastic damping may be added to the constitutive relation (5) by incorporating the Kelvin–Voigt viscoelastic model which then becomes

$$\left(1-\mu_0^2\frac{\partial^2}{\partial x^2}\right)t_{r\theta} = \left(1-l_s^2\frac{\partial^2}{\partial x^2}\right)G\left(\varepsilon_{r\theta}+g\dot{\varepsilon}_{r\theta}\right) = \left(1+g\frac{\partial}{\partial t}\right)\left(1-l_s^2\frac{\partial^2}{\partial x^2}\right)G\varepsilon_{r\theta} \tag{6}$$

where g is the damping coefficient and $\dot{\varepsilon}_{r\theta} = \partial\varepsilon_{r\theta}/\partial t$ is the rate of shear strain with respect to the time variable t.

3 EQUATION OF MOTION OF SIZE-DEPENDENT RODS

The displacement field in a rod of volume V, length L and cross-sectional area A takes the following form:

$$\theta_1 = \theta\left(x,t\right), \quad \theta_2 = 0, \quad \theta_3 = 0 \tag{7}$$

Here θ_1, θ_2 and θ_3 denote the time dependent rotations about the x, y and z directions, respectively. The shear strain of a torsional rod and its gradient with respect to x can be, respectively, written as

$$\varepsilon_{r\theta} = r\frac{\partial\theta}{\partial x}, \quad \varepsilon_{r\theta,x} = r\frac{\partial^2\theta}{\partial x^2} \tag{8}$$

The strain energy, U, after integrating by parts and using Eq. (8) is then

$$U = \int_V \left(\sigma_{r\theta}\varepsilon_{r\theta} + \sigma^{(1)}_{r\theta x}\varepsilon_{r\theta,x}\right) dV = \int_0^L T\frac{\partial\theta}{\partial x}dx + \left[T^{(1)}\frac{\partial\theta}{\partial x}\right]_0^L \tag{9}$$

where T and $T^{(1)}$ are stress resultants of, respectively, the total stress and the higher-order stress which are given below in addition to the stress resultant of the classical stress $T^{(0)}$

$$T = \int_A rt_{r\theta}dA, \quad T^{(1)} = \int_A r\sigma^{(1)}_{r\theta x}dA, \quad T^{(0)} = \int_A r\sigma_{r\theta}dA \tag{10}$$

Considering the torsional motion of the rod and its velocity gradient, the kinetic energy, K, can be written as

$$K = \frac{1}{2}\rho J\int_0^L \left(\frac{\partial\theta}{\partial t}\right)^2 dx + \frac{1}{2}\rho J l_k^2\int_0^L \left(\frac{\partial^2\theta}{\partial x\partial t}\right)^2 dx \tag{11}$$

where ρ is the density of the rod and l_k is the kinetic material length-scale parameter associated with the velocity gradient. The surrounding medium is assumed to be a Winkler type model, where k_{EM} is the linear torsional stiffness. Then, the external work done by the surrounding medium is

$$W = -\int_0^L k_{EM}\theta^2 d\theta \tag{12}$$

The equation of motion is obtained by applying Hamilton's Principle and the fundamental lemma of calculus variations. After integration by parts with respect to t as well as x, and setting the initial conditions to zero, the following equation of motion can be derived:

$$-\rho J\frac{\partial^2\theta}{\partial t^2} + \rho J l_k^2\frac{\partial^4\theta}{\partial x^2\partial t^2} + \frac{\partial T}{\partial x} - k_{EM}\theta = 0 \tag{13}$$

After some mathematical manipulations, the expression of T in terms of the rotation as

$$T = GJ\left(\frac{\partial\theta}{\partial x} + g\frac{\partial^2\theta}{\partial x\partial t}\right) - GJl_s^2\left(\frac{\partial^3\theta}{\partial x^3} + g\frac{\partial^4\theta}{\partial x^3\partial t}\right) + \mu_0^2\rho J\left(\frac{\partial^3\theta}{\partial x\partial t^2} - l_k^2\frac{\partial^5\theta}{\partial x^3\partial t^2}\right) + \mu_0^2 k_{EM}\frac{\partial\theta}{\partial x} \tag{14}$$

Furthermore, differentiating this expression for T with respect to x and substituting into Eq. (13) gives the equation of motion in terms of the rotation as

$$\rho J\left[-\frac{\partial^2\theta}{\partial t^2} + l_k^2\frac{\partial^4\theta}{\partial x^2\partial t^2} + \mu_0^2\left(\frac{\partial^4\theta}{\partial x^2\partial t^2} - l_k^2\frac{\partial^6\theta}{\partial x^4\partial t^2}\right)\right] + GJ\left[\frac{\partial^2\theta}{\partial x^2} + g\frac{\partial^3\theta}{\partial x^2\partial t} - l_s^2\left(\frac{\partial^4\theta}{\partial x^4} + g\frac{\partial^5\theta}{\partial x^4\partial t}\right)\right] - k_{EM}\left(\theta - \mu_0^2\frac{\partial^2\theta}{\partial x^2}\right) = 0 \tag{15}$$

This governing equation of motion for θ is subjected to the following classical and non-classical boundary conditions specified at each of the ends $x = 0$ and $x = L$:

$$T = 0 \quad \text{or} \quad \theta = 0 \tag{16}$$

$$T^{(1)} = 0 \quad \text{or} \quad \frac{\partial\theta}{\partial x} = 0 \tag{17}$$

Performing further mathematical manipulations leads to the expressions of the stress resultants of the classical stress $T^{(0)}$ and the stress gradient $T^{(1)}$ in terms of the rotation

$$T^{(0)} = \frac{\mu_0^4}{\mu_0^2 - l_s^2}\left[k_{EM}\frac{\partial\theta}{\partial x} + \rho J\left(\frac{\partial^3\theta}{\partial x\partial t^2} - l_k^2\frac{\partial^5\theta}{\partial x^3\partial t^2}\right)\right] - GJ\frac{\mu_0^2 l_s^2}{\mu_0^2 - l_s^2}\left(\frac{\partial^3\theta}{\partial x^3} + g\frac{\partial^4\theta}{\partial x^3\partial t}\right) + GJ\left(\frac{\partial\theta}{\partial x} + g\frac{\partial^2\theta}{\partial x\partial t}\right) \tag{18}$$

$$T^{(1)} = \frac{\mu_0^4 l_s^2}{\mu_0^2 - l_s^2}\left[k_{EM}\frac{\partial^2\theta}{\partial x^2} + \rho J\left(\frac{\partial^4\theta}{\partial x^2\partial t^2} - l_k^2\frac{\partial^6\theta}{\partial x^4\partial t^2}\right)\right] - GJ\frac{\mu_0^2 l_s^4}{\mu_0^2 - l_s^2}\left(\frac{\partial^4\theta}{\partial x^4} + g\frac{\partial^5\theta}{\partial x^4\partial t}\right) + GJl_s^2\left(\frac{\partial^2\theta}{\partial x^2} + g\frac{\partial^3\theta}{\partial x^2\partial t}\right) \tag{19}$$

Using the following non-dimensional parameters:

$$\xi = \frac{x}{L}, \quad \tau = \frac{t}{L}\sqrt{\frac{G}{\rho}}, \quad \theta(x,t) = \theta(\xi,\tau)$$

$$\hat{\mu_0} = \frac{\mu_0}{L}, \quad \hat{g} = \frac{g}{L}\sqrt{\frac{G}{\rho}}, \quad \hat{l}_s = \frac{l_s}{L}, \quad \hat{l}_k = \frac{l_k}{L}, \quad \hat{k}_{EM} = \frac{k_{EM}L^2}{GJ}, \tag{20}$$

the governing equations, Eq. (15), along with the associated boundary conditions, Eqs. (16) and (17), can be written in non-dimensional form as

$$
\begin{aligned}
-\frac{\partial^2\theta}{\partial\tau^2} + \hat{l}_k^2\frac{\partial^4\theta}{\partial\xi^2\partial\tau^2} + \hat{\mu}_0^2\left(\frac{\partial^4\theta}{\partial\xi^2\partial\tau^2} - \hat{l}_k^2\frac{\partial^6\theta}{\partial\xi^4\partial\tau^2}\right) + \\
\frac{\partial^2\theta}{\partial\xi^2} + \hat{g}\frac{\partial^3\theta}{\partial\xi^2\partial\tau} - \hat{l}_s^2\left(\frac{\partial^4\theta}{\partial\xi^4} + \hat{g}\frac{\partial^5\theta}{\partial\xi^4\partial\tau}\right) - \hat{k}_{EM}\left(\theta - \hat{\mu}_0^2\frac{\partial^2\theta}{\partial\xi^2}\right) = 0
\end{aligned}
\tag{21}
$$

subject to the boundary conditions specified at each of the ends $\xi = 0$ and $\xi = 1$

$$\hat{T} = 0 \quad \text{or} \quad \theta = 0 \tag{22a}$$

$$\hat{T}^{(1)} = 0 \quad \text{or} \quad \frac{\partial\theta}{\partial\xi} = 0 \tag{22b}$$

The non-dimensional expressions of the stress resultants T, $T^{(0)}$ and $T^{(1)}$ are given by

$$
\begin{aligned}
\hat{T} = \hat{\mu}_0^2 k_{EM}\frac{\partial\theta}{\partial\xi} + \hat{\mu}_0^2\left(\frac{\partial^3\theta}{\partial\xi\partial\tau^2} - \hat{l}_k^2\frac{\partial^5\theta}{\partial\xi^3\partial\tau^2}\right) - \\
\hat{l}_s^2\left(\frac{\partial^3\theta}{\partial\xi^3} + \hat{g}\frac{\partial^4\theta}{\partial\xi^3\partial\tau}\right) + \left(\frac{\partial\theta}{\partial\xi} + \hat{g}\frac{\partial^2\theta}{\partial\xi\partial\tau}\right)
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
\hat{T}^{(0)} = \frac{\hat{\mu}_0^4}{\hat{\mu}_0^2 - \hat{l}_s^2}\left[k_{EM}\frac{\partial\theta}{\partial\xi} + \left(\frac{\partial^3\theta}{\partial\xi\partial\tau^2} - \hat{l}_k^2\frac{\partial^5\theta}{\partial\xi^3\partial\tau^2}\right)\right] - \\
\frac{\hat{\mu}_0^2\hat{l}_s^2}{\hat{\mu}_0^2 - \hat{l}_s^2}\left(\frac{\partial^3\theta}{\partial\xi^3} + \hat{g}\frac{\partial^4\theta}{\partial\xi^3\partial\tau}\right) + \left(\frac{\partial\theta}{\partial x} + \hat{g}\frac{\partial^2\theta}{\partial\xi\partial\tau}\right)
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
\hat{T}^{(1)} = \frac{\hat{\mu}_0^4}{\hat{\mu}_0^2 - \hat{l}_s^2}\left(\hat{k}_{EM}\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^4\theta}{\partial\xi^2\partial\tau^2} - \hat{l}_k^2\frac{\partial^6\theta}{\partial\xi^4\partial\tau^2}\right) - \\
\frac{\hat{\mu}_0^2 l_s^2}{\hat{\mu}_0^2 - \hat{l}_s^2}\left(\frac{\partial^4\theta}{\partial\xi^4} + \hat{g}\frac{\partial^5\theta}{\partial\xi^4\partial\tau}\right) + \hat{l}_s^2\left(\frac{\partial^2\theta}{\partial\xi^2} + \hat{g}\frac{\partial^3\theta}{\partial\xi^2\partial\tau}\right)
\end{aligned}
\tag{25}
$$

4 ANALYTICAL SOLUTION OF THE VIBRATION PROBLEM

The solution of the non-dimensional Partial Differential Equation (PDE) (21) proceeds in the usual way by assuming a separable solution of the form $\theta(\xi,\tau) = \phi(\xi)e^{st}$. The resulting Ordinary Differential Equation (ODE) for $\phi(\xi)$ can be written as

$$\alpha(s)\frac{\mathrm{d}^4\phi}{\mathrm{d}\xi^4} + \beta(s)\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + \gamma(s)\phi = 0 \tag{26}$$

where

$$\alpha(s) = s^2\hat{l}_k^2\hat{\mu}_0^2 + \hat{g}s\hat{l}_s^2 + \hat{l}_s^2, \beta(s) = -\left(s^2\hat{l}_k^2 + s^2\hat{\mu}_0^2 + \hat{k}_{EM}\hat{\mu}_0^2 + \hat{g}s + 1\right), \gamma(s) = s^2 + \hat{k}_{EM} \tag{27}$$

Since Eq. (26) is a linear ODE for fixed s, the solution is of the form $\phi(\xi) = Ce^{\lambda\xi}$, which gives a quadratic equation in λ^2 as

$$\alpha(s)\lambda^4 + \beta(s)\lambda^2 + \gamma(s) = 0 \tag{28}$$

which has four solutions $\lambda_i(s)$ for $i = 1, \ldots, 4$.

Three sets of boundary conditions are now considered for the torsional rod, where the mode shapes may be determined directly. These special cases are (a) Clamped Forcing-Clamped Forcing (CF-CF), (b) Clamped Forcing - Free Strained (CF-FS), (c) Free Strained - Free Strained (FS-FS). Table 1 gives the mode shapes, where C_n is an arbitrary constant, and it is straightforward to verify that these functions satisfy the given boundary conditions by directly substituting into the expressions for $\hat{T}$ and $\hat{T}^{(1)}$ given earlier. Here we will assume that $\hat{\mu}_0 \neq \hat{l}_s$ for the CF-CF and FS-FS cases, since $\hat{T}^{(1)}$ given by Eq. (25) is not defined when $\hat{\mu}_0 = \hat{l}_s$; the solution for $\hat{\mu}_0 = \hat{l}_s$ is considered in detail in the next section. The mode shapes in Table 1 give direct expressions for λ_i, which may be substituted into Eq. (28), using the expressions for α, β and γ from Eq. (27), to give the following quadratic equation for s:

$$as^2 + bs + c = 0 \tag{29}$$

where the expressions of a, b and c for the three boundary conditions CF-CF, CF-FS and FS-FS are given in Table 2. This quadratic equation is easily solved to obtain the solutions for s, and hence the corresponding natural frequencies and damping ratios.
For each set of boundary conditions, solutions for particular cases of interest can be obtained, namely, local undamped ($\hat{\mu}_0 = \hat{l}_k = \hat{l}_s = \hat{g} = 0$), local damped ($\hat{\mu}_0 = \hat{l}_k = \hat{l}_s = 0$) and asymptotic cases. In the last case, the asymptotic frequencies are obtained by taking the asymptotic expansion of Eq. (29) when $n \to \infty$ and then keeping the leading terms of order n^4 before solving for s to give the natural frequencies and damping ratios. It is worth noting that the solutions for the CF-CF and FS-FS boundary conditions are identical because the ordinary differential equation (26) contains $\frac{\mathrm{d}^4\phi}{\mathrm{d}\xi^4}$ and $\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}$ and the mode shapes for these boundary conditions are, respectively, given by $\sin(n\pi\xi)$ and $\cos(n\pi\xi)$.

5 FORMULATION AND SOLUTION FOR PARTICULAR CASE $\hat{\mu}_0 = \hat{l}_s$

The partial differential equation for θ given in Eq. (21) is well defined for $\hat{\mu}_0 = \hat{l}_s$, and is given by

$$\begin{aligned} -\frac{\partial^2\theta}{\partial\tau^2} + \hat{l}_k^2\frac{\partial^4\theta}{\partial\xi^2\partial\tau^2} + \frac{\partial^2\theta}{\partial\xi^2} + \hat{g}\frac{\partial^3\theta}{\partial\xi^2\partial\tau} - \hat{k}_{EM}\theta + \\ \hat{\mu}_0^2\left(\frac{\partial^4\theta}{\partial\xi^2\partial\tau^2} - \hat{l}_k^2\frac{\partial^6\theta}{\partial\xi^4\partial\tau^2} - \frac{\partial^4\theta}{\partial\xi^4} - \hat{g}\frac{\partial^5\theta}{\partial\xi^4\partial\tau} + \hat{k}_{EM}\frac{\partial^2\theta}{\partial\xi^2}\right) = 0 \end{aligned} \tag{30}$$

This differential equation can be conveniently written as

$$\left(1 - \hat{\mu}_0^2\frac{\partial^2}{\partial\xi^2}\right)\mathcal{L}(\theta) = 0 \tag{31}$$

where

$$\mathcal{L}(\theta) = -\frac{\partial^2\theta}{\partial\tau^2} + \hat{l}_k^2\frac{\partial^4\theta}{\partial\xi^2\partial\tau^2} + \frac{\partial^2\theta}{\partial\xi^2} + \hat{g}\frac{\partial^3\theta}{\partial\xi^2\partial\tau} - \hat{k}_{EM}\theta \tag{32}$$

Table 1: Boundary conditions (BC) and modeshapes for the analytical solution.

BC	BC Equations	Mode shape	λ_n^2
CF-CF	$\phi(0)=0,\ \hat{T}^{(1)}(0)=0,$	$\phi(\xi)=C_n\sin(n\pi\xi)$	$-n^2\pi^2$
	$\phi(1)=0,\ \hat{T}^{(1)}(1)=0$	$n\geq 1$	
CF-FS	$\phi(0)=0,\ \hat{T}^{(1)}(0)=0,$	$\phi(\xi)=C_n\sin\left(\frac{2n-1}{2}\pi\xi\right)$	$-\left(n-\frac{1}{2}\right)^2\pi^2$
	$\frac{d\phi(1)}{d\xi}=0,\ \hat{T}(1)=0$	$n\geq 1$	
FS-FS	$\frac{d\phi(0)}{d\xi}=0,\ \hat{T}(0)=0,$	$\phi(\xi)=C_n\cos(n\pi\xi)$	$-n^2\pi^2$
	$\frac{d\phi(1)}{d\xi}=0,\ \hat{T}(1)=0$	$n\geq 0$	

Table 2: Expressions of the constants of the characteristic polynomial asociated with the analytical solution.

Case	Polynomial constants
CF-CF & FS-FS	$a = n^4\pi^4\hat{l}_k^2\hat{\mu}_0^2+n^2\pi^2\hat{l}_k^2+n^2\pi^2\hat{\mu}_0^2+1$
	$b = \hat{g}n^4\pi^4\hat{l}_s^2+\hat{g}n^2\pi^2$
	$c = n^4\pi^4\hat{l}_s^2+n^2\pi^2\hat{k}_{EM}\hat{\mu}_0^2+n^2\pi^2+\hat{k}_{EM}$
CF-FS	$a = \left(n-\frac{1}{2}\right)^4\pi^4\hat{l}_k^2\hat{\mu}_0^2+\left(n-\frac{1}{2}\right)^2\pi^2\left(\hat{l}_k^2+\hat{\mu}_0^2\right)+1$
	$b = \hat{g}\left(n-\frac{1}{2}\right)^4\pi^4\hat{l}_s^2+\hat{g}\left(n-\frac{1}{2}\right)^2\pi^2$
	$c = \left(n-\frac{1}{2}\right)^4\pi^4\hat{l}_s^2+\left(n-\frac{1}{2}\right)^2\pi^2\left(\hat{k}_{EM}\hat{\mu}_0^2+1\right)+\hat{k}_{EM}$

However, the expressions for the classical stress and the stress gradient resultants given in non-dimensional form in Eqs. (24) and (25) are not defined when $\hat{\mu}_0 = \hat{l}_s$. In this case, it can be shown that the total stress resultant given by (23) degenerates to

$$\hat{T} = \left(\frac{\partial \theta}{\partial \xi} + \hat{g} \frac{\partial^2 \theta}{\partial \xi \partial \tau} \right) \tag{33}$$

Consider now the calculation of the higher order stress $\hat{T}^{(1)}$ when $\hat{\mu}_0 = \hat{l}_s$. This cannot be calculated from Eq. (25), because the denominator term $\hat{\mu}_0^2 - \hat{l}_s^2$ is zero. To determine $\hat{T}^{(1)}$, the solution proceeds in the usual way by assuming a separable solution of the form $\theta(\xi, \tau) = \phi(\xi) e^{st}$, which gives a linear fourth order ODE for $\phi(\xi)$. The general solution is

$$\phi(\xi) = C_1 e^{\lambda(s)\xi} + C_2 e^{-\lambda(s)\xi} + C_3 e^{\hat{\mu}_0 \xi} + C_4 e^{-\hat{\mu}_0 \xi} \tag{34}$$

where λ is obtained from

$$\lambda^2 \left(\hat{l}_k^2 s^2 + 1 + \hat{g} s \right) - s^2 - \hat{k}_{EM} = 0 \tag{35}$$

After few mathematical manipulations, it can be shown that the second equation of (10) can be written as

$$\hat{T}^{(1)}(\xi) = (1 + \hat{g} s)\, \hat{\mu}_0^2 \left(\frac{C_1 e^{\lambda \xi} + C_2 e^{-\lambda \xi}}{\lambda^2 - \hat{\mu}_0^2} - \frac{C_3 \xi e^{\hat{\mu}_0 \xi} - C_4 \xi e^{-\hat{\mu}_0 \xi}}{2 \hat{\mu}_0^3} \right) \tag{36}$$

This gives the relationship between the C_i coefficients to implement boundary conditions for $\hat{T}^{(1)}$ when $\hat{\mu}_0 = \hat{l}_s$. For the three boundary conditions CF-CF, CF-FS and FS-FS, and using the same mode shapes as in the general case, the eigenvalues s, are the solutions of the quadratic equation in Eq. (29), where expressions for a, b and c are given in Table 3.

To show the consistency of the solution, Figs. 1 and 2 illustrate the variation of the first three frequencies and damping ratios as a function of $\Delta\mu l_s = \hat{l}_s - \hat{\mu}_0$ for CF-CF/FS-FS and CF-FS boundary conditions, respectively. When $\hat{\mu}_0$ is close to $\hat{l}_s$, the solution is computed based the equation of motion (21). For the particular case where $\hat{\mu}_0 = \hat{l}_s$, the solution is computed based on the equation of motion (31) and is shown with the symbol $\circ$ in Figs. 1 and 2. It is evident from these figures that the frequency and damping ratio solutions are continuous and do not show any sign of inconsistency.

6 CONCLUSIONS

A combined nonlocal strain and velocity gradient theory was used to study the size-dependent torsional vibration of a viscoelastic nanorod embedded in an elastic medium. The equation of motion and the related boundary conditions were derived using the Hamiltonian principle. Frequencies and damping ratios were obtained for different classical and non-classical boundary conditions. The case where the strain gradient and nonlocal parameters are equal ($l_s = \mu_0$) may seem to result in an inconsistency to the general equation of motion and the related non-classical boundary conditions. A study of this case was treated thoroughly in this paper demonstrating that the proposed solution would not show any inconsistency.

Table 3: Expressions of the constants of the characteristic polynomial and the corresponding eigenvalues for the particular case $\hat{\mu}_0 = \hat{l}_s$.

Case	Polynomial constants	Eigenvalue solution
CF-CF & FS-FS	$a = n^2\pi^2 {l_k}^2 + 1$ $b = gn^2\pi^2$ $c = n^2\pi^2 + k_{EM}$	$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
CF-FS	$a = \left(n - \frac{1}{2}\right)^2 \pi^2 l_k^2 + 1$ $b = g\left(n - \frac{1}{2}\right)^2 \pi^2$ $c = \left(n - \frac{1}{2}\right)^2 \pi^2 + k_{EM}$	$\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

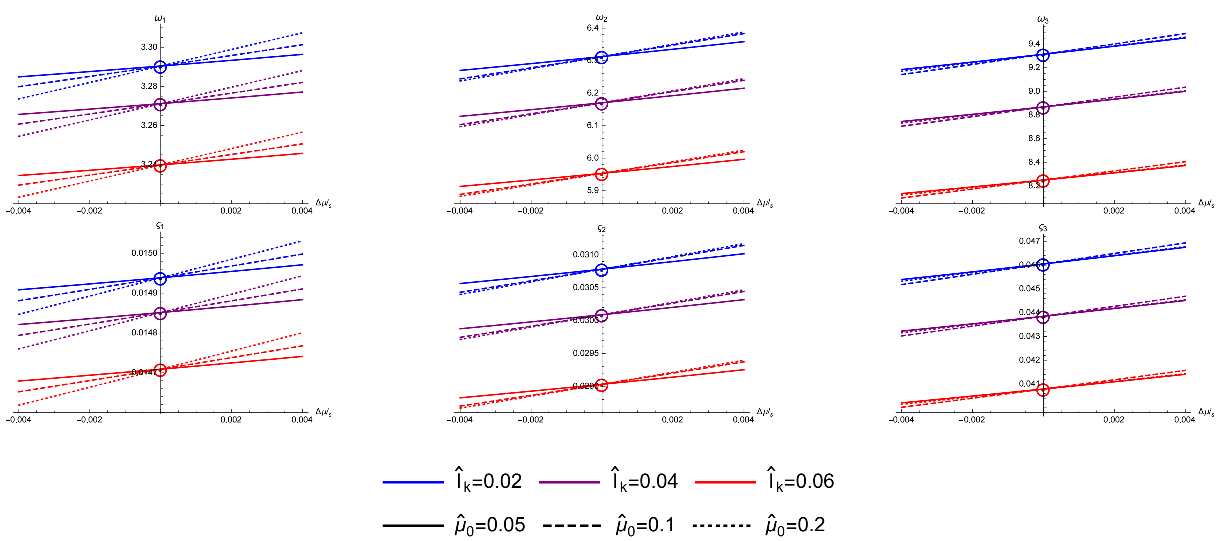

Figure 1: Variation of the first three frequencies and damping ratios as a function of $\Delta\mu l_s = \hat{l}_s - \hat{\mu}_0$ for CF-CF / FS-FS boundary conditions (ie, when $\hat{\mu}_0$ is equal or close to $\hat{l}_s$); $\hat{k}_{EM} = 1$; Symbol $\circ$ in the plot is obtained from solution of particular case $\hat{\mu}_0 = \hat{l}_s$ given in Table 2.

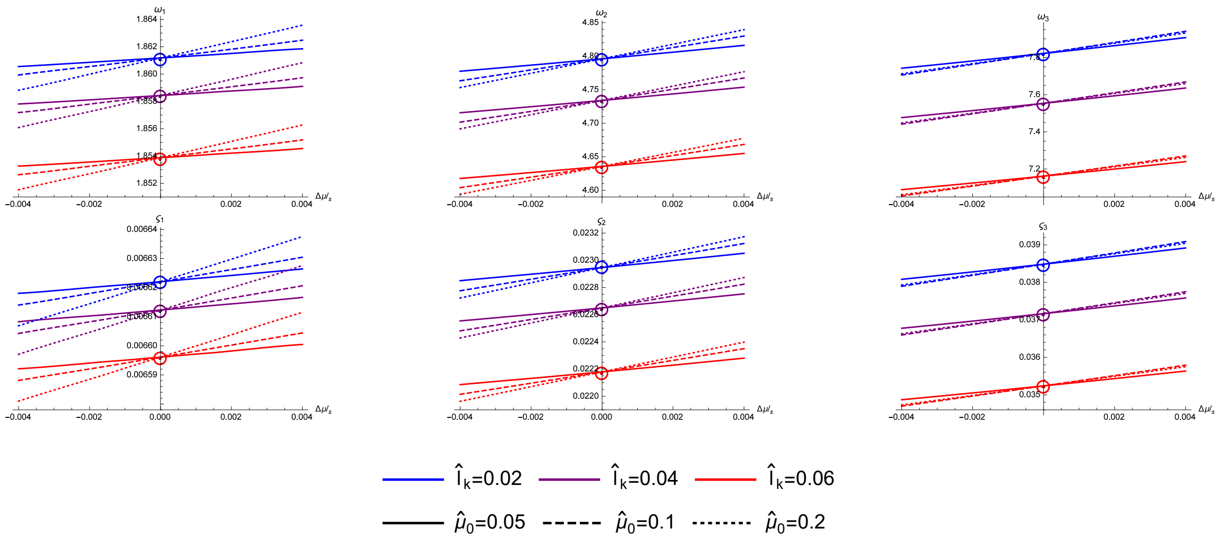

Figure 2: Variation of the first three frequencies and damping ratios as a function of $\Delta\mu l_s = \hat{l}_s - \hat{\mu}_0$ for CF-FS boundary conditions (ie, when $\hat{\mu}_0$ is equal or close to $\hat{l}_s$); $\hat{k}_{EM} = 1$; Symbol $\circ$ in the plot is obtained from solution of particular case $\hat{\mu}_0 = \hat{l}_s$ given in Table 2.

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